Mediterranean Youth Mathematical Championship (MYMC)
Trieste, July 8, 2015

Morning round

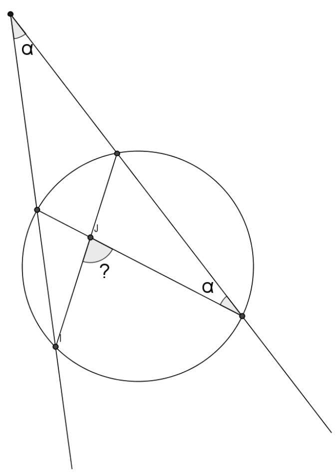
WE1
(Leonardo Pisano, *Liber Abbaci*, 1202)
There were two men: the first had 12 fish, while the second had 13. The fish all were equally priced. From the first man the customs agent took away 1 fish and 12 denari. From the second man the customs agent took 2 fish, and gave the man back 7 denari. Find the price of each fish in denari.

A) Less than 22
B) 22
C) Between 22 and 23
D) 23
E) More than 23

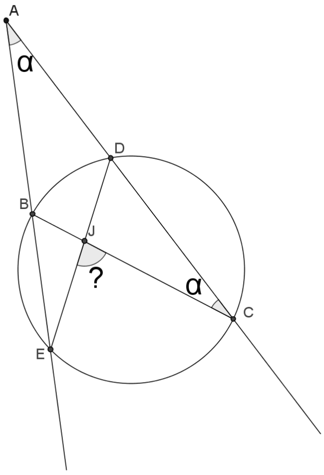
Solution
Let *f* stand for fish and *d* for denari. If $1f + 12d$ is the tax for 12 fish, then $(13/12)f + 13d$ should be the tax for 13 fish. Hence $2f – 7d$ must equal $(13/12)f + 13d$. It follows that $(11/12)f$ equals $20d$ and the price of each fish is $(12/11)·20d = (240/11)d$. Rearranging, we find that the answer is A).

WE2
Two angles of the same measure α are shown in the figure; therefore, the angle marked '?' is equal to:

A) α
B) 2 α
C) 3 α
D) 4 α
E) 5 α



Solution



The angle $\angle EBC$, which is an external angle of the triangle ABC, is equal to 2α; therefore the angle $\angle EDC$ is also equal to 2α. The unknown angle is an external angle of DJC, therefore it is equal to $\alpha + 2\alpha = 3\alpha$. The answer is C).

WE3

(Leonardo Pisano, *Liber Abbaci*, 1202)

There is a vat which has four different holes. The vat empties at a different rate depending on which hole is opened: through the first hole it takes 1 day; through the second it takes 2 days; through the third it takes 3 days; and through the fourth it takes 4 days. If all four holes are opened at the same time, how many hours will it take for the vat to empty?

A) More than 13 hours
B) 13 hours
C) Between 12 and 13 hours
D) 12 hours
E) Less than 12 hours

Solution

Let V be the volume of the vat and let x be the number of hours required. Then $x(V/24) + x(V/48) + x(V/72) + x(V/96)$ must equal V, i.e. $x/24 + x/48 + x/72 + x/96 = 1$. Hence $25x = 288$, therefore $x=288/25$. The answer is E).

WE4

The product of two positive numbers x and y is 3 times their sum; the same product is equal to 6 times the difference between the two numbers. We assume $x \geq y$. What are the values of x and y?

Solution

We know that $xy = 3(x + y) = 6(x - y)$. From the latter equality we find that $x = 3y$; therefore $3y^2 = 3\cdot 4y$. Remembering that $y \neq 0$, we conclude that $x = 12$; $y = 4$.

WE5

Joseph had his 31st birthday on the 1st January 2015, and he noticed that 2015 is a multiple of 31. Supposing that Joseph lives until he is 100 years old, how many other times will his age be a divisor of the year?

Solution

The answer is 3. If Joseph turned 31 in 2015, then he was born in 1984. We want to know for which x (with $31 < x \leq 100$) is x a divisor of $(1984 + x)$. This is true if and only if x is a divisor of 1984. As $1984 = 31 \cdot 2^6$, we conclude that Joseph's age will be a divisor of the year when he is 32, 62, or 64, i.e. in 2016, 2046, and 2048.

WE6

Let Q be a square with two of its vertices on one face of a cube, and its other two vertices on the opposite face of the cube. The cube has edges of length 1. What is the maximum value of the square of the perimeter of Q? [We ask for the value of (perimeter of Q)^2.]

Solution

The answer is 18. Due to symmetry, the sides of Q lying on the two opposing faces of the cube must be parallel to one of the diagonals of these two faces. Let x be the distance from any vertex of Q to the nearest vertex of the cube (with $0 < x < 1 - (1/2)\cdot\sqrt{2}$). As Q is a square, we find that $1 + (\sqrt{2}\,x)^2 = 2\cdot(1 - x)^2$, meaning that $1 + 2\,x^2 = 2 + 2\,x^2 - 4\,x$ and so $x = 1/4$. Therefore, Q has side length $\ell = \sqrt{2}\,(1 - x) = (3/4)\cdot\sqrt{2}$ and so the square of its perimeter will be $[4\cdot(3/4)\cdot\sqrt{2}\,]^2 = 9\cdot 2 = 18$.

WE7

The 24 inhabitants of a village on a distant island are of two kinds: the knaves, who always lie, and the knights, who always tell the truth.

The first inhabitant says: ‘In the village, the number of knaves is a multiple of 1.’

The second inhabitant says: ‘In the village, the number of knaves is a multiple of 2.’

The third inhabitant says: ‘In the village, the number of knaves is a multiple of 3.’

And so on until the final inhabitant, who says: ‘In the village, the number of knaves is a multiple of 24.’

(We note that the inhabitants do not consider the number 0 as a multiple of other numbers, while every positive number is considered a multiple of itself.)

How many of the inhabitants are knaves?

Solution

The answer is 18. Let $d(n)$ be the number of divisors of n: we need to solve the equation $n + d(n) = 24$. This is because for a population with n knaves, the other $24 - n$ inhabitants (and only they) speak the truth. Therefore, there are $24 - n$ true statements, and so n must have $24 - n$ divisors. The function $d(n)$ has some simple properties: for example, if p is prime, then $d(p) = 2$, while if p and q are two distinct primes, then $d(p\cdot q) = 4$. We also note that $d(n) < n$ for every $n > 2$. With these properties in mind, we find that the only solution is 18. In fact, 18 has 6 divisors, which are 1, 2, 3, 6, 9, 18.

WE8

A figure is composed of two right-angled triangles which share two vertices, such that the hypotenuse of the first triangle is a cathetus of the second (and the triangles have no other points in common). The triangles are similar and have integer sides.
What is the area of the smallest figure which satisfies these conditions?

Solution

The answer is 204. Firstly, let ka and kb be the lengths of the catheti of the smaller triangle, and kc be the length of its hypotenuse, with (a, b, c) a primitive Pythagorean triple (that is, a, b and c are positive relatively prime integers, and $c^2 = a^2 + b^2$). Let us suppose that in their similarity, kc corresponds to kb: then the sides of the bigger triangle are kc, kca / b, and kc^2 / b. It is clear that for these numbers to be integers it is necessary that k be a multiple of b, i.e. $k = hb$. Therefore, the total area of the figure is: $(1/2)\ (hab\ hb^2 + hbc\ hca) = (h^2\ ab)/2\ (2\ b^2 + a^2)$.
For any choice of a, b, c, the minimum area is clearly obtained when $h = 1$. Using the well-known triple (3, 4, 5) we obtain an area of 204; since there are no other (primitive) triples with at least one cathetus of length smaller than 4, any other triple will give us a total area greater than $(3/2)\ 4^4 = 384$. Therefore the minimum area is 204.

WE9

An urn contains 7 balls, 3 of which are white and 4 of which are black.
All of the balls are taken out of the urn, one at a time and at random, and are placed in a row in the order in which they are removed.
What is the probability that the first two white balls in the row are consecutive?

A) 1/4
B) 1/3
C) 3/7
D) 1/2
E) 4/7

Solution

The answer is C).
The number of possible outcomes is 7 choose 3. The number of successful outcomes is 6 choose 2. For if we have a sequence with two Ws and four Bs, we can substitute the first W with ‘WW’: in this way we find all of the required sequences. We conclude that the probability is therefore 15/35 = 3/7.

WE10

Let A = {490, 497, 506, 512}.
Determine how many of the elements of A divide the number $N = 46^6 – 25^6 + 39^6 – 32^6$.

A) 0
B) 1
C) 2
D) 3
E) 4

Solution

The answer is B).

Neither 5 nor 11 divides N, since $N \equiv 1^6 - 0^6 + (-1)^6 - 2^6 \equiv 3 \pmod{5}$ and $N \equiv 2^6 - 3^6 + 6^6 - (-1)^6 \equiv -2 + 8 + 5 - 1 \equiv -1 \pmod{11}$; it therefore follows that neither $490 = 5 \cdot 98$ nor $506 = 11 \cdot 46$ divides N.

Then, we note that N can be considered as the sum of 3 terms: the first is $46^6 = 64 \cdot 23^6 = 64 \cdot D$, with D an odd number, which is not divisible by 128; the second is $39^6 - 25^6 = (39+25)\cdot(39-25)\cdot(39^4 + 39^2 \cdot 25^2 + 25^4) = 128 \times M$, where M is an integer, which is clearly a multiple of 128; and the third is 32^6, which is also a multiple of 128. Therefore 128 does not divide their sum, N, and $512 = 128 \cdot 4$ cannot divide N.

Finally, it is easy to see that 497 divides N. Firstly, $497 = 7 \cdot 71$. We can also see that $14 = 46 - 32 = 39 - 25$ divides both $46^6 - 32^6$ and $39^6 - 25^6$, while $71 = 46 + 25 = 39 + 32$ divides both $46^6 - 25^6$ and $39^6 - 32^6$, and since 7 and 71 are coprime, this means that their product divides N.