Mediterranean Youth Mathematical Championship (MYMC)
Trieste, July 8, 2015

Afternoon round – First stage

RE1A

Let *ABC* be an equilateral triangle of side length 8. It is divided into 64 smaller equilateral triangles, each of side length 1, by a set of straight lines parallel to the sides of *ABC*. How many equilateral triangles of any size appear in the figure?

A) 64
B) 128
C) 156
D) 170
E) 192

Solution

The answer is D), as one can see directly keeping in mind that some of the smaller triangles can be turned upside down (∇, not just Δ).

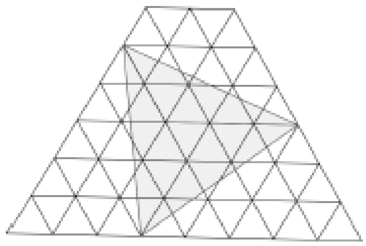
In general one can say the following.

Consider an equilateral triangle of side n divided as in the text. For $n = 1$ there is obviously only one triangle in the picture; for $n = 2$ there are five (four of them of side length 1, and one of side length 2). Let $T(n)$ be the answer for a triangle of side length n. Now take an equilateral triangle of side length $n + 1$ and fix one of its sides, s. The number of equilateral triangles in this triangle that do *not* have a vertex on side s is obviously $T(n)$. We now need to add the triangles which have two vertices on s – we have one such triangle for every pair of distinct division points on s, or $(n+2)(n+1) / 2$ triangles altogether – and the triangles with just one vertex on s: n of side length 1, $n - 2$ of side length 2, and so on. Hence, $T(n+2) - T(n) = T(n+2) - T(n+1) + T(n+1) - T(n) = (n+3)(n+2) / 2 + [(n+1) + (n-1) + \ldots] + (n+2)(n+1) / 2 + [n + (n-2) + \ldots] = (n+3)(n+2) / 2 + (n+2)(n+1) / 2 + (n+2)(n+1) / 2$.

From this we can easily compute $T(4) = 27$, $T(6) = 78$, $T(8) = 170$.

RE1B

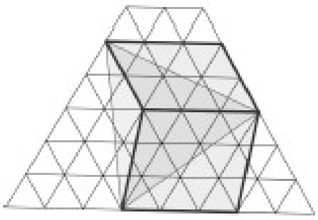
The following figure is made up of 60 congruent equilateral triangles, each of area 1. Calculate the area of the grey triangle.



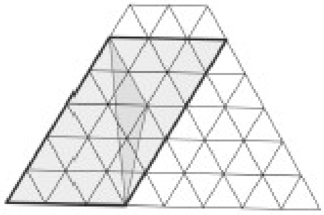
Solution

The answer is 19.

The two following figures illustrate one way of solving the problem, using the areas of half parallelograms.



6 + 9 = 15



15 – 8 – 3 = 4

GE1A

For every integer n, the number $n^9 - n$ is divisible by

A) 4
B) 22
C) 30
D) 34
E) 51

Solution

The answer is C).

It is clear that $n^9 - n = n\,(n^8 - 1) = (n - 1)\,n\,(n + 1)\,(n^2 + 1)\,(n^4 + 1)$ is divisible by 2 and 3 as it contains the three consecutive factors $(n - 1)\,n\,(n + 1)$. The remaining three factors assure divisibility by 5 if $n = 5k$, $n = 5k + 1$, $n = 5k + 4$ for integer k. For the remaining two cases we turn to the factor $(n^2 + 1)$: for $n = 5k + 2$ we have $(5k + 2)^2 + 1 = 25k^2 + 20k + 5 = 5\,(5k^2 + 4k + 1)$, while for $n = 5k + 3$ we have $(5k + 3)^2 + 1 = 25k^2 + 30k + 10 = 5\,(5k^2 + 6k + 2)$.

GE1B

The equation $x^2 + a\,x + b + 1 = 0$ (where a, b are integers) has two distinct integer roots which are different from 0. Therefore, necessarily:

A) $a^2 + b^2$ is the square of an integer;
B) $a^2 + b^2$ is not prime;
C) at least one of the numbers a, b is not prime;
D) $a + b$ is always even;
E) $a + b$ is always odd.

Solution

The answer is B).

Let x_1 and x_2 be the distinct roots of the equation $x^2 + a\,x + b + 1 = 0$; we find that $a = -(x_1 + x_2)$ and $b + 1 = x_1 \cdot x_2$.

Therefore $a^2 + b^2 = (x_1 + x_2)^2 + (x_1 x_2 - 1)^2 = x_1^2 + x_2^2 + x_1^2 x_2^2 + 1 = (x_1^2 + 1) \cdot (x_2^2 + 1)$.