Mediterranean Youth Mathematical Championship (MYMC)
Rome, July 18, 2013

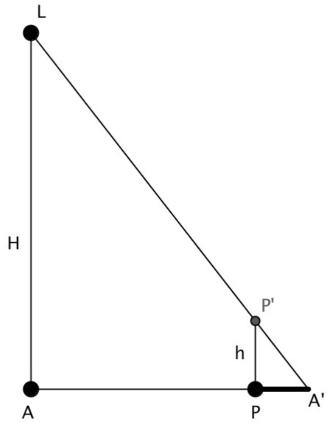
Afternoon round – Second stage

RE2A

Let there be a rectangular field *ABCD* that is lit by *4* lights based in the vertices of the rectangle, all at the same distance from the floor (like a football field during a nocturnal match), and let us consider a player standing in the field at position *P*. Let *PA'* be the player's shadow caused by the light based at point *A*, and similarly let *PB'*, *PC'*, *PD'* be the other three shadows.
If the position *P* of the player changes, we can say that:

A) The area of the quadrilateral *A'B'C'D'* changes, but its angles don't
B) The area of the quadrilateral *A'B'C'D'* does not change, but its angles do
C) Neither the area nor the angles of the quadrilateral *A'B'C'D'* change
D) The quadrilateral *A'B'C'D'* is always a parallelogram and it is a rectangle only when the player is in the middle of the field
E) The quadrilateral *A'B'C'D'* is always a parallelogram and it is a rhombus only when the player is in the middle of the field

Solution



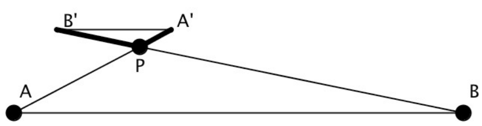
Let *H = AL* be the distance of the light *L* from the floor, *h* the height of the player *PP'*. As the player is standing (i.e. is vertical) the segment *PP'* is parallel to the segment *AL*, meaning the triangles *LAA'* and *P'PA'* are similar. It follows that

$$\frac{x_A}{h} = \frac{x_A + d_A}{H}$$

where x_A=*PA'* is the length of the shadow caused by the light at *L* and d_A=*PA* is the distance of the player from the point *A*.
Rearranging, we find that

$$\frac{x_A}{d_A} = \frac{h}{H-h}$$

and so, given that this ratio does not depend on the position of the player and the light considered, we have *PA':PA = PB':PB*



and therefore the triangles *APB* and *A'PB'*, which have the same angle at *P*, are similar. It follows that *A'B'* is parallel to *AB* and

$$\frac{A'B'}{AB} = \frac{h}{H-h}$$

Applying the same reasoning to the sides *BC*, *CD* and *DA* of the rectangle we find that *B'C'* is parallel to *BC*, *C'D'* is parallel to *CD*, *D'A'* is parallel to *DA* and the sides of the small rectangle are shrunk by the same ratio which depends only upon the height of the *4* lights (which, by assumption, is the same) and upon the height of the player, which does not change when the player moves. The correct answer is therefore C).

RE2B

Two runners run for one hour at a constant speed on a *400* m track: the first runs at *10* km/h, the second at *7* km/h. The two runners start together at the same point. How many times do they cross each other (after the start) if they are running in opposite directions?

Solution

It is simpler to consider a single runner who runs at the speed of *17* km/h, while the other runner remains stationary at the starting point: in fact, the number of times in which they meet depends solely on the difference between the two speeds: $10 - (-7) = 17$ km/h.

In one hour, the runner will cover *17* km; if the track is *400* m long, the runner will cross over the starting point (encountering the other runner) *42* times, because $17000 : 400 = 42.5$.

GE2A

There is an urn containing white and black balls; the number n of white balls is equal to the number of black balls. There is also a second urn containing white and black balls, but the number w of white balls is greater than the number b of black balls. Find values of n, b, w (greater than *1*) such that: if two balls are taken out simultaneously, the probability that these two balls are the same colour is the same should the two balls be taken from the first urn or from the second urn.

(If the problem has no solution, write $n = 0$, $b = 0$, $w = 0$; if there is more than one possible solution, write only one solution.)

Solution

The statement is equivalent to the following equation:

$$\frac{2\binom{n}{2}}{\binom{2n}{2}} = \frac{\binom{w}{2}+\binom{b}{2}}{\binom{w+b}{2}}$$

Solutions can be found (with $w>b$) by trial and error.

Alternatively, we consider the triangular numbers t_k; if we set

$w = t_{k+1} + 1$

$b = t_k + 1$

$n = w \cdot b$

the equation is satisfied (the proof is a little long, but simple).

For example:	$w = 2$	$b = 1$	$n = 2$
	$w = 7$	$b = 4$	$n = 28$
	$w = 11$	$b = 7$	$n = 77$

etc.

GE2B

The number *2010* is divisible by *2* but not by *4*, is divisible by *3* but not by *9*, and is divisible by *5* but not by *25*. What is the smallest integer greater than *2010* which shares the same properties?

Solution

It is clear that to *2010* we have to add a multiple of *2*, *3*, and *5*, that is a multiple of *30*. We have that:

$2010 + 30 = 2040$ is a multiple of *4*

$2010 + 60 = 2070$ is a multiple of *9*

$2010 + 90 = 2100$ is a multiple of *25*

$2010 + 120 = 2130$ is the answer.